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FFT, DA, and Mori-Tanaka approximation to determine the elastic moduli of three-phase composites with the random inclusions

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Abstract. In this work, some solutions such as Mori-Tanaka approximation (MTA), Differential approximations (DA), and Fast Fourier transformation method (FFT) were applied to estimate the elastic bulk and shear modulus of three-phase composites in 2D. In which two different sizes of circular inclusions are arranged randomly non-overlapping in a continuous matrix. The numerical solutions using FFT analysis were compared with DA, MTA, and Hashin-Strikman's bounds. The MTA and DA reasonably agreeable solution with the FFT solution shows the effectiveness of the approximation methods, which makes MTA, DA useful with simplicity and ease of application.

Keywords: Elastic modulus / Fast Fourier transformation method (FFT) / Mori-Tanaka approximation / differential approximations / composite materials

1 Introduction

Investigations on the macroscopic properties of multiphase materials and media have been started with the classical works of Maxwell, Voigt, Reuss, Einstein [1], and continued with the bounds of Hill [2], and Hashin and Shtrikman [3], Pham & Nguyen [4, 5]. Further studies aim to construct better estimates by including more detailed information about the microstructure of the materials. Approximation schemes have been constructed based on microscopic models. Based on the classical work of Eshelby [6] on the elastic ellipsoidal inclusion problem, effective medium approximations have been developed to predict the overall behavior of the general non-dilute composites such as approximation methods (Christensen [7], Mori and Tanaka [8], and the recent Polarization approximation [9]), in which construction of approximate formulae that can describe the macroscopic property accurately over ranges of volume proportions of component materials for engineering uses. Strong numerical methods such as the Finite Element and Fast Fourier ones have been developed and used effectively. Numerical homogenization techniques determining the effective properties give reliable results but challenge engineers by computational costs, especially in the case of complex microstructure. The Fast Fourier Transform has been used to compute the effective properties of periodic composites by Michel et al. [10], Moulinec [11], Bonnet [12] in the context of elasticity. However, most of these studies were applied to two-phase material composites with simple periodic structures such as square or hexagonal models. In this work, some solutions are proposed to determine the macro-elastic modulus for 3-phase composite materials in 2D (or transverse isotropic unidirectional fiber-reinforced composite) with a more complex structure than previous studies [13]. In Section 2, the Mori-Tanaka approximation for elastic moduli is introduced with the explicit expressions of the estimates for the three-phase composite with random two different phase inclusions. In Section 3, the DA approximation for elastic bulk and shear modulus are developed, HS bounds are presented in Section 4. In Section 5, the FFT method is developed to directly calculate the effective moduli of three-phase composites with the random distribution of fiber inclusions. After that (Sect. 6), numerical examples will be presented to compare the results of MTA, DA, and Hashin-Shtrikman bounds and the FFT method.

2 Mori-Tanaka approximation (MTA) for elastic modulus

Consider n-component transversely isotropic unidirectional elastic composites of randomly oriented inclusions of type α ($\alpha = 2, ..., n$). The matrix phase has the volume fraction v_M and the α-inclusion has the volume fraction $v_{I\alpha}$. The bulk

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modulus and shear modulus of the matrix are K_M and μ_M, respectively, those of the α inclusion phases are $K_{I\alpha}$ and $\mu_{I\alpha}$. The MTA, derived as an approximate solution to the field equations for the composite to compute the elastic bulk modulus K_{MTA} and shear modulus μ_{MTA} has the expressions [1,8]

$$K_{MTA} = \frac{v_M K_M + \sum_{\alpha=2}^{n} v_{I\alpha} K_{I\alpha} D_{K\alpha}(K_{I\alpha}, \mu_{I\alpha}, K_M, \mu_M)}{v_M + \sum_{\alpha=2}^{n} v_{I\alpha} D_{K\alpha}(K_{I\alpha}, \mu_{I\alpha}, K_M, \mu_M)} \quad (1)$$

$$\mu_{MTA} = \frac{v_M \mu_M + \sum_{\alpha=2}^{n} v_{I\alpha} \mu_{I\alpha} D_{K\alpha}(K_{I\alpha}, \mu_{I\alpha}, K_M, \mu_M)}{v_M + \sum_{\alpha=2}^{n} v_{I\alpha} D_{K\alpha}(K_{I\alpha}, \mu_{I\alpha}, K_M, \mu_M)}. \quad (2)$$

$D_{K\alpha}$, $D_{\mu\alpha}$ are functions depending on the inclusion-shape, $D_{K\alpha}$, $D_{\mu\alpha}$ with α-circular inclusion phases in 2D are specified

$$D_{K\alpha}(K_{I\alpha}, \mu_{I\alpha}, K_M, \mu_M) = \frac{K_M + \mu_M}{K_{I\alpha} + \mu_M} \quad (3)$$

$$D_{\mu\alpha}(K_{I\alpha}, \mu_{I\alpha}, K_M, \mu_M) = \frac{\mu_M + \frac{K_M \mu_M}{K_M + 2\mu_M}}{\mu_{I\alpha} + \frac{K_M \mu_M}{K_M + 2\mu_M}}, \alpha = 1, 2. \quad (4)$$

The three-component composite that are two circular inclusions having elastic bulk modulus K_{I1}, K_{I2} shear modulus μ_{I1}, μ_{I2} and volume fraction v_{I1}, v_{I2} in a matrix having the elastic moduli K_M, μ_M and volume fraction v_M. In the case of three-component matrix composites, the bulk modulus K and the elastic shear modulus μ formula of Mori-Tanaka approximation can be written as:

See equations (5)–(6) below

where the dilute suspension expression $D_{K\alpha}$, $D_{\mu\alpha}$ for an inclusion α has been defined in (3), (4). Equations (5) and (6) will be used to determine the elastic moduli of three-phase composites.

3 Differential approximations (DA)

Consider n-phase suspension of randomly oriented inclusions of type α ($\alpha = 2, \ldots, n$), with elastic bulk modulus $K_{I\alpha}$, shear modulus $\mu_{I\alpha}$ (volume proportion $v_{I\alpha}$) in a matrix of elastic moduli K_M, μ_M (volume proportion v_M). The differential scheme construction process for the suspension starts with the base matrix phase M. At each step of the procedure, we add proportionally infinitesimal volume amounts $v_{I\alpha}\Delta$t ($\Delta t \ll 1$, $\alpha = 2, \ldots, n$) of randomly oriented inclusions into already constructed composite of the previous step, which contains volume fractions $v_{I\alpha}t$ of the inclusion phases (the parameter t increases from 0 to 1, as the differential scheme proceeds). The newly added particles will see an effective continuum, owing to their relative sizes, and the new composite can be considered as a dilute suspension of particles from phases α, of volume fractions

$$\frac{v_{I\alpha}\Delta t}{1 + \sum_{\alpha=2}^{n} v_{I\alpha}\Delta t} = \frac{v_{I\alpha}\Delta t}{1 + v_I \Delta t} \quad (7)$$

where $v_I = \sum_{\alpha=2}^{n} v_{I\alpha}$, in a matrix of elastic bulk modulus K, shear modulus μ (v_I is the total volume fractions of the included phases). The elastic modulus of the new composite is

$$K + dK = K + \sum_{\alpha=2}^{n} \frac{v_{I\alpha}\Delta t}{1 + v_I \Delta t}(K_{I\alpha} - K) D_{K\alpha}(K_{I\alpha},\, \mu_{I\alpha},\, K,\, \mu) \quad (8)$$

$$\mu + d\mu = \mu + \sum_{\alpha=2}^{n} \frac{v_{I\alpha}\Delta t}{1 + v_I \Delta t}(\mu_{I\alpha} - \mu) D_{\mu\alpha}(K_{I\alpha},\, \mu_{I\alpha},\, K,\, \mu) \quad (9)$$

where the dilute suspension expression $D_{K\alpha}$, $D_{\mu\alpha}$ for an inclusion α has been defined in (3), (4). Since the volume fraction of the included phase α increases by

$$v_{I\alpha}dt = \frac{v_{I\alpha}t + v_{I\alpha}\Delta t}{1 + v_I \Delta t} - v_{I\alpha}t = \frac{v_{I\alpha}\Delta t}{1 + v_{I\alpha}\Delta t}(1 - v_I t) \quad (10)$$

$$K_{MTA} = \frac{v_M K_M + v_{I1} K_{I1} D_{K1}(K_{I1}, \mu_{I1}, K_M, \mu_M) + v_{I2} K_{I2} D_{K2}(K_{I2}, \mu_{I2}, K_M, \mu_M)}{v_M + v_{I1} D_{K1}(K_{I1}, \mu_{I1}, K_M, \mu_M) + v_{I2} D_{K2}(K_{I2}, \mu_{I2}, K_M, \mu_M)} \quad (5)$$

$$\mu_{MTA} = \frac{v_M \mu_M + v_{I1} \mu_{I1} D_{\mu 1}(K_{I1}, \mu_{I1}, K_M, \mu_M) + v_{I2} \mu_{I2} D_{\mu 2}(K_{I2}, \mu_{I2}, K_M, \mu_M)}{v_M + v_{I1} D_{\mu 1}(K_{I1}, \mu_{I1}, K_M, \mu_M) + v_{I2} D_{\mu 2}(K_{I2}, \mu_{I2}, K_M, \mu_M)} \quad (6)$$

we obtain the following differential equation for the elastic bulk modulus K, shear modulus μ of the composite

$$\frac{dK}{dt} = \frac{1}{1 - v_I \Delta t} \sum_{\alpha=2}^{n} v_{I\alpha}(K_{I\alpha} - K) D_{K\alpha}(K_{I\alpha}, \mu_{I\alpha}, K, \mu) \quad (11)$$

$$\frac{d\mu}{dt} = \frac{1}{1 - v_I \Delta t} \sum_{\alpha=2}^{n} v_{I\alpha}(K_{I\alpha} - K) D_{K\alpha}(K_{I\alpha}, \mu_{I\alpha}, K, \mu) \quad (12)$$

$$K(0) = K_M, \ \mu(0) = \mu_M, \ 0 \leq t \leq 1. \quad (13)$$

Differential equations (11) and (12) will be used to determine the elastic moduli of three-phase composites. Though the above construction process of differential scheme corresponds to certain idealistic hierarchical models formed on widely-separated scales, the approximation aims at usual multiphase suspensions of inclusions in a matrix.

4 Hashin-Strikman bounds

HS bounds on the effective elastic moduli of isotropic d-dimensional composites can be presented as [3]
– Elastic bulk modulus

$$P_k\left(\frac{2(d-1)}{d}\mu_{\min}\right) \leq K^{eff} \leq P_k\left(\frac{2(d-1)}{d}\mu_{\max}\right) \quad (14)$$

where

See equation (15) below.

– Shear modulus

$$P_\mu(\mu^*(K_{\min}, \mu_{\min})) \leq \mu^{eff} \leq P_\mu(\mu^*(K_{\max}, \mu_{\max})) \quad (16)$$

where

See equation (17) below.

5 FFT simulation for three-phase composites

The FFT method uses the classical expansion along with the Neuman series of the solution of the periodic elastic problem in Fourier space, based on the Green's tensor and exact expressions of the shape factors in Fourier space [11,12,14,15]. In this section, the FFT method is briefly presented for calculating the effective elastic moduli of three-component materials in 2D.

Behavior of the component materials is described by Hooke' law:

$$\boldsymbol{\sigma}(\mathrm{x}) = \mathbf{C}(\mathrm{x}) : \boldsymbol{\varepsilon}(\mathrm{x}) \quad (18)$$

where $\boldsymbol{\sigma}(\mathrm{x})$ and $\boldsymbol{\varepsilon}(\mathrm{x})$ are respectively the local stress and strain fields, the stress field satisfies the equilibrium condition

$$\nabla . \boldsymbol{\sigma}(x) = 0. \quad (19)$$

Let x denote the position of a point in the unit cell. $\mathbf{C}(\mathrm{x})$ is the fourth order local elastic tensor of the heterogeneous medium, one is given by

$$\mathbf{C}(\mathbf{x}) = \sum_{\alpha} C_\alpha I_\alpha(\mathbf{x}), \ I_\alpha(\mathbf{x}) = \begin{cases} 1, \mathbf{x} \in V_\alpha \\ 0, \mathbf{x} \notin V_\alpha \end{cases} \quad (20)$$

α designates the phase ($\alpha = I_1 ; I_2$ or M).

We shall denote the Fourier transform of a V-periodic function $\mathbf{F}(\mathrm{x})$ of cartesian $\mathbf{x}(x_1, \ x_2, \ x_3)$ as $\hat{\mathrm{F}}(\xi)$

$$\mathbf{F}(\mathbf{x}) = \sum_{\xi} \hat{\mathrm{F}}(\xi) e^{i\xi.\mathbf{x}}, \quad \hat{\mathrm{F}}(\xi) = \langle \mathbf{F}(\mathbf{x}) e^{-i\xi.\mathbf{x}} \rangle = \frac{1}{V} \int_V \mathbf{F}(\mathbf{x}) e^{-i\xi.\mathbf{x}} d\mathbf{x} \quad (21)$$

with $\xi(\xi_1, \ \xi_2, \ \xi_3)$ being the wave vector, the symbol "*" designates the product of convolution $\xi = \xi_k \mathbf{e}_k, \ \xi_j = \frac{n_j \pi}{a_j}, \ (n_j = 0, \pm 1, \pm 2, ..., \pm \infty), \ j = 1, 2, 3.$

$$\begin{aligned} &P_\mu(\mu_{*\min}) \leq \mu^{eff} \leq P_\mu(\mu_{*\max}) \quad , \quad \mu_*(k, \mu) = \mu \frac{d^2 k + 2(d+1)(d-2)\mu}{2dk + 4d\mu} \\ &P_k(k_*) = \left(\sum_\alpha \frac{v_\alpha}{k_\alpha + k_*}\right)^{-1} - k_* \quad , \quad P_\mu(\mu_*) = \left(\sum_\alpha \frac{v_\alpha}{\mu_\alpha + \mu_*}\right)^{-1} - \mu_* \\ &k_{\min} = \min\{k_1, \ldots, k_N\}, \quad k_{\max} = \max\{k_1, \ldots, k_N\}, \\ &\mu_{\min} = \min\{\mu_1, \ldots, \mu_N\}, \quad \mu_{\max} = \max\{\mu_1, \ldots, \mu_N\}. \end{aligned} \quad (15)$$

$$\begin{aligned} &P_\mu(\mu^*) = \left(\sum_\alpha \frac{v_\alpha}{\mu_\alpha + \mu^*}\right)^{-1} - \mu^*, \mu^*(K, \mu) = \mu \frac{d^2.K + 2(d+1)(d-2)\mu}{2d.K + 4d.\mu}, \\ &\mu_{\min} = \min\{\mu_1, \ldots, \mu_n\}, \ \mu_{\max} = \max\{\mu_1, \ldots, \mu_n\} \end{aligned} \quad (17)$$

The Fourier transformation of elastic tensor is

$$C(\xi) = \int_V C(\mathrm{x}) e^{-i\xi.\mathbf{x}} d\mathrm{x} = \sum_\alpha C_\alpha I_\alpha(\xi) \quad (22)$$

where $I_\alpha(\xi)$ are the shape functions, defined by

$$I_\alpha(\xi) = \frac{1}{V} \int_{V_\alpha} e^{i\xi.x} dV. \quad (23)$$

In the case of circle-inclusion, the function $I_\alpha(\xi)$ is given by Nemat-Nasser [16])

$$Ia(\xi) = 2S_{I\alpha} \frac{J_1(\eta)}{\eta} .e^{i\xi.\mathbf{x}_c(\alpha)} \quad (24)$$

where $\eta_1 = R(\xi_1^2 + \xi_2^2)^{1/2}$, $S_{I\alpha} = \pi R^2$, R is the radii of circle-inclusion, $\mathbf{x}_c(\alpha)$ is the vector position of the center of the inclusion α ; and ξ_1, ξ_2 are the components of ξ; J_1 is the Bessel function of first kind and first order. $I_M(\xi)$ can be derived from relation

$$\sum_\alpha I_\alpha(\xi) = 0, \ \ \forall \xi \neq 0. \quad (25)$$

For $\xi = 0$, one have $I_\alpha(0) = {}^{V_\alpha}/_{V}$.

Substituting the Fourier transformation of the local stress, strain fields into equilibrium equation (19), the problem in a unit cell is solved by explicit recurrence process in Fourier space. That can be rewritten in the form [10,11]

$$\begin{cases} \hat{\varepsilon}^{i+1}(\xi) = \hat{\varepsilon}^{i}(\xi) - \hat{\Gamma}(\xi) : \sum_\alpha \mathbf{C}_\alpha I_\alpha(\xi) * \hat{\varepsilon}^{i}(\xi), & \xi \neq 0 \\ \hat{\varepsilon}^{i+1} = \mathbf{E}^0, \quad \xi = 0 \end{cases} \quad (26)$$

in which $\hat{\boldsymbol{\sigma}}(\xi)$ and $\hat{\boldsymbol{\varepsilon}}(\xi)$ are respectively Fourier transformation of $\boldsymbol{\sigma}(\mathrm{x})$ and $\boldsymbol{\varepsilon}(\mathrm{x})$, $\hat{\Gamma}(\xi)$ is the Greens's tensor, the symbol "*" designates the product of convolution. The value of the Green's tensor is given for an isotropic reference medium by Mura [1].

$$\hat{\Gamma}_{iiii} = 4A\frac{\xi_i^2}{\xi^2} - B\frac{\xi_i^4}{\xi^4} \quad (27)$$

$$\hat{\Gamma}_{iijj} = -4B\frac{\xi_i^2\xi_j^2}{\xi^4} \quad (28)$$

$$\hat{\Gamma}_{iiij} = 2A\frac{\xi_i\xi_j}{\xi^2} - B\frac{\xi_i^3\xi_j}{\xi^4} \quad (29)$$

$$\hat{\Gamma}_{1212} = A - B\frac{\xi_1^2\xi_2^2}{\xi^4}, \ \hat{\Gamma}_{3232} = A\frac{\xi_2^2}{\xi^2}, \ \hat{\Gamma}_{3131} = A\frac{\xi_1^2}{\xi^2}, \ \hat{\Gamma}_{3132} = A\frac{\xi_1\xi_2}{\xi^2} \quad (30)$$

where $A = \frac{1}{4}\mu$, $B = \frac{\lambda+\mu}{\mu(\lambda+2\mu)}$, λ and μ are Lame coefficients.

Relationship between $\hat{\boldsymbol{\sigma}}(\xi)$ and $\hat{\boldsymbol{\varepsilon}}(\xi)$ is described by expression

$$\hat{\boldsymbol{\sigma}}(\xi) = \mathbf{C}(\xi) * \hat{\varepsilon}(\xi). \quad (31)$$

For the three-component medium considered, the expression (26) can be written as

See equation (32) below.

At convergence of the iterative process, one finds

$$\boldsymbol{\sigma}(\xi = 0) = \mathbf{C}^{eff} * \mathbf{E}^0. \quad (33)$$

The numerical algorithm is given as follows

Iteration i=1: $\hat{\varepsilon}^1(\xi) = 0 \quad \forall \xi \neq 0$; $\quad \hat{\varepsilon}^1(0) = \mathbf{E}^0$

$$\hat{\boldsymbol{\sigma}}^1(\xi) = \mathbf{C}(\xi) * \hat{\boldsymbol{\varepsilon}}^1(\xi).$$

Iteration i: $\hat{\boldsymbol{\varepsilon}}^i(\xi)$ and $\hat{\boldsymbol{\sigma}}^i(\xi)$ are known

$$\hat{\boldsymbol{\varepsilon}}^{i+1}(\xi) = \hat{\varepsilon}^i(\xi) - \hat{\Gamma}^0(\xi) : \hat{\boldsymbol{\sigma}}^i(\xi)$$

$$\hat{\boldsymbol{\sigma}}^{i+1}(\xi) = \mathbf{C}(\xi) * \hat{\boldsymbol{\varepsilon}}^{i+1}(\xi).$$

The convergence of the iterative procedure is reached when $\frac{\|\hat{\boldsymbol{\sigma}}^{i+1}(\xi) - \hat{\boldsymbol{\sigma}}^{i}(\xi)\|}{\|\hat{\boldsymbol{\sigma}}^{i}(\xi)\|} < \varepsilon$, where ε is a prescribed value ($\varepsilon = 10^{-3}$).

6 Applications

In this section, we use the FFT method, MTA approximation, and DA approximation to estimate the effective elastic moduli of the elastically-isotropic composites 2D.

We consider 3 examples, with

(A): $K_M = 2$, $\mu_M = 1$, $K_{I1} = 100$, $\mu_{I1} = 60$, $K_{I2} = 20$, $\mu_{I2} = 10$

(B): $K_M = 30$, $\mu_M = 16$, $K_{I1} = 10$, $\mu_{I1} = 6$, $K_{I2} = 1$, $\mu_{I2} = 0.5$

(C): $K_M = 30$, $\mu_M = 16$, $K_{I1} = 4$, $\mu_{I1} = 2$, $K_{I2} = 100$, $\mu_{I2} = 60$.

We consider three-phase composites, in which two different size balls are arranged randomly non-overlapping (Fig. 1). For numerical FFT illustrations, 60 circles inclusion were planted randomly in a unit cell by Matlab program such that there is no circle overlapping, in which a unit cell having the dimension $L = 1$ along each space direction containing inclusion (Fig. 1, left), the minimum

$$\begin{cases} \hat{\varepsilon}^{i+1}(\xi) = \hat{\varepsilon}^{i}(\xi) - \hat{\Gamma}(\xi) : \left[C_M : \hat{\varepsilon}^{i}(\xi) + (C_{I1} - C_M)(I_{I1}(\xi) * \hat{\varepsilon}^{i}(\xi)) + (C_{I2} - C_M)(I_{I2}(\xi) * \hat{\varepsilon}^{i}(\xi)) \right], \ \xi \neq 0 \\ \hat{\varepsilon}^{i+1} = \mathbf{E}^0, \quad \xi = 0 \end{cases} \quad (32)$$

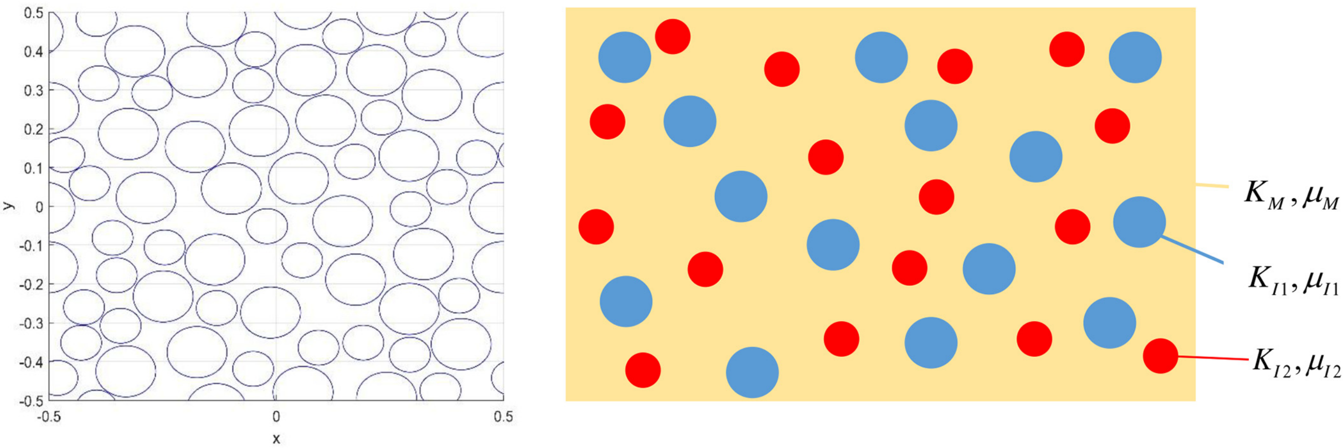

Fig. 1. Unit cell containing 60 circle-inclusion randomly palaced (left), three phase model of composite with two different circles inclusion (right).

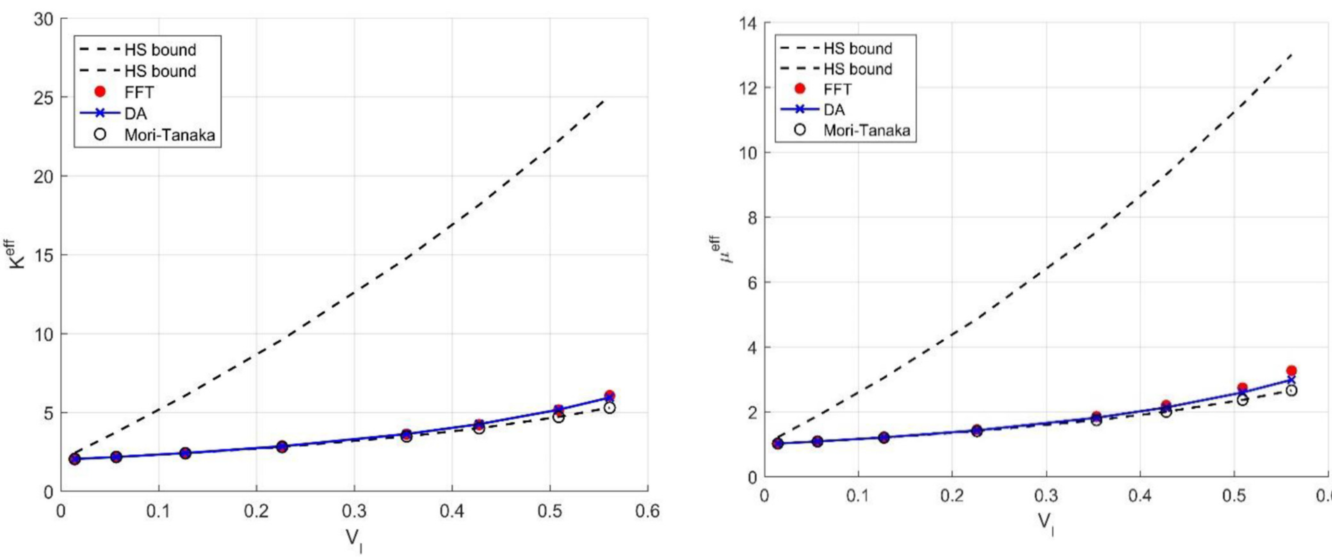

Fig. 2. Elastic bulk (left) and shear modulus (right) of three-phase composites with the case (A).

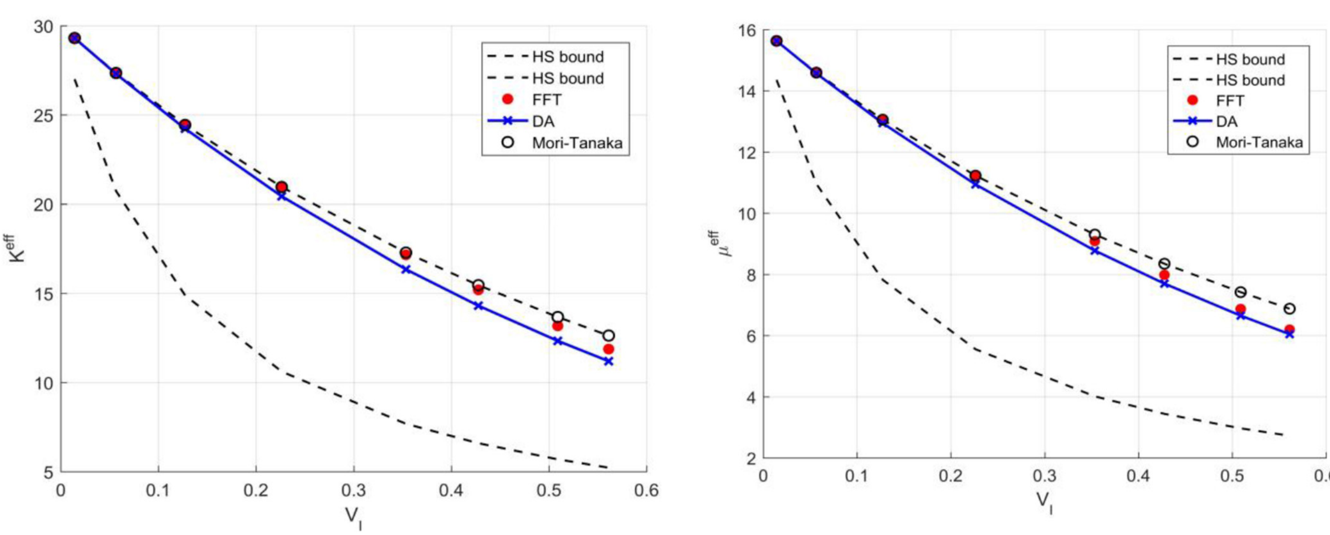

Fig. 3. Elastic bulk (left) and shear modulus (right) of three-phase composites with the case (B).

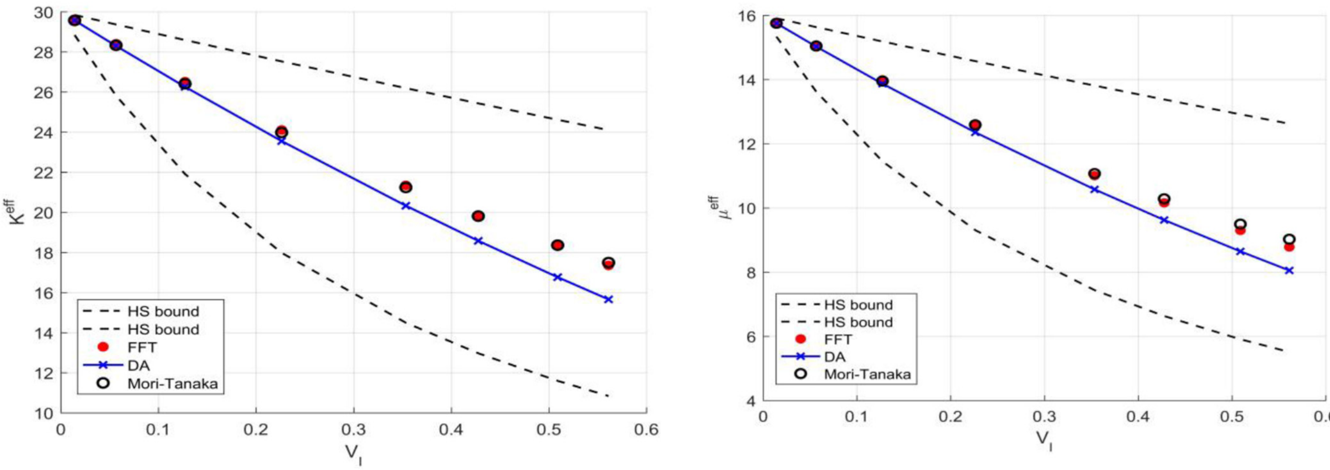

Fig. 4. Elastic bulk (left) and shear modulus (right) of three-phase composites with the case (C).

Table 1. Comparison of results (K^{eff}) of FFT, MTA, DA, and HS bound for the case (A), $R_1 = \sqrt{2}R_2$.

R_1	$v_I = v_{I1} + v_{I2}$	HS_Lower	FFT	MTA	AD	HS_Upper
0.01	0.0141	2.0401	2.0019	2.0401	2.0402	2.4266
0.02	0.0565	2.1670	2.1905	2.1670	2.1690	3.7422
0.03	0.1272	2.4039	2.4446	2.4039	2.4158	6.0626
0.04	0.2262	2.8020	2.8722	2.8020	2.8481	9.6103
0.05	0.3534	3.4749	3.6133	3.4749	3.6235	14.7730
0.055	0.4276	3.9902	4.2142	3.9902	4.2501	18.1543
0.06	0.5089	4.7102	5.1462	4.7102	5.1663	22.2267
0.063	0.5611	5.2929	6.0566	5.2929	5.9368	25.0714

Table 2. Comparison of results (μ^{eff}) of FFT, MTA, DA, and HS bounds for the case (A), $R_1 = \sqrt{2}R_2$.

R_1	$v_I = v_{I1} + v_{I2}$	HS_Lower	FFT	MTA	AD	HS_Upper
0.01	0.0141	1.0201	1.0009	1.0201	1.0202	1.2139
0.02	0.0565	1.0838	1.0957	1.0838	1.0848	1.8755
0.03	0.1272	1.2028	1.2251	1.2028	1.2087	3.0493
0.04	0.2262	1.4028	1.4485	1.4028	1.4260	4.8609
0.05	0.3534	1.7413	1.8545	1.7413	1.8163	7.5346
0.055	0.4276	2.0009	2.1967	2.0009	2.1323	9.3102
0.06	0.5089	2.3642	2.7370	2.3642	2.5952	11.4750
0.063	0.5611	2.6568	3.2650	2.6568	2.9853	13.0046

distance of is 0.01. In our calculations, a grid 128×128 is considered. The FFT result simulation is obtained from the algorithm in Section 4. The FFT results compared with Differential approximation, Mori-Tanaka approximation, and Hashin–Shtrikman bounds over ranges of $v_I = v_{I1} + v_{I2}$, $v_{I1} = 2v_{I2}$ (all the inclusions in one phase have the same size, the dimensionless radius R_{I1} varies from 0.01 to 0.063 and R_{I2} varies from 0.0071 to 0.0445) are reported in Figures 2–4 and Tables 1–6. Numerical FFT results and MTA, DA approximations are quite close and converge at small volume fractions of inclusion phases and diverge at large proportions of suspended particles, all the results fall inside the Hashin-Shtrikman' bounds, as expected. The DA, MTA approximations are asymptotically exact at dilute suspensions of included particles, but become inevitably less so good at higher proportions of included phases. At large proportions of included phases, the details of particles' interactions of particular micro-structures should be accounted for more accurate estimations.

Table 3. Comparison of results (K^{eff}) of FFT, MTA, DA, and HS bounds for the case (B), $R_1 = \sqrt{2}R_2$.

R_1	$v_I = v_{I1} + v_{I2}$	HS_Lower	FFT	MTA	AD	HS_Upper
0.01	0.0141	27.0008	29.3346	29.3073	29.3045	29.3073
0.02	0.0565	20.7361	27.3857	27.3490	27.3013	27.3490
0.03	0.1272	14.8922	24.4704	24.4361	24.2275	24.4361
0.04	0.2262	10.6114	20.9567	20.9591	20.4373	20.9591
0.05	0.3534	7.6847	17.1532	17.2799	16.3414	17.2799
0.055	0.4276	6.5947	15.1940	15.4534	14.3044	15.4534
0.06	0.5089	5.6915	13.1736	13.6700	12.3320	13.6700
0.063	0.5611	5.2239	11.8714	12.6282	11.1937	12.6282

Table 4. Comparison of results (μ^{eff}) of FFT, MTA, DA, and HS bound for the case (B), $R_1 = \sqrt{2}R_2$.

R_1	$v_I = v_{I1} + v_{I2}$	HS_Lower	FFT	MTA	AD	HS_Upper
0.01	0.0141	14.3571	15.6492	15.6325	15.6310	15.6325
0.02	0.0565	10.9579	14.6187	14.5954	14.5693	14.5954
0.03	0.1272	7.8257	13.0759	13.0572	12.9430	13.0572
0.04	0.2262	5.5546	11.1903	11.2284	10.9405	11.2284
0.05	0.3534	4.0132	9.0875	9.3016	8.7766	9.3016
0.055	0.4276	3.4413	7.9843	8.3483	7.6984	8.3483
0.06	0.5089	2.9685	6.8710	7.4195	6.6516	7.4195
0.063	0.5611	2.7241	6.1943	6.8779	6.0456	6.8779

Table 5. Comparison of results (K^{eff}) of FFT, MTA, DA, and HS bound for the case (C), $R_1 = \sqrt{2}R_2$.

R_1	$v_I = v_{I1} + v_{I2}$	HS_Lower	FFT	MTA	AP	HS_Upper
0.01	0.0141	28.8402	29.6093	29.5712	29.5695	29.8412
0.02	0.0565	25.8158	28.4026	28.3316	28.3002	29.3683
0.03	0.1272	21.9081	26.5124	26.4089	26.2583	28.5910
0.04	0.2262	17.9787	24.1064	23.9813	23.5522	27.5253
0.05	0.3534	14.4934	21.3444	21.2405	20.3287	26.1921
0.055	0.4276	12.9700	19.8680	19.8086	18.5774	25.4331
0.06	0.5089	11.5948	18.3266	18.3615	16.7648	24.6169
0.063	0.5611	10.8379	17.3536	17.4929	15.6576	24.1013

Table 6. Comparison of results (μ^{eff}) of FFT, MTA, DA, and HS bound for the case (C), $R_1 = \sqrt{2}R_2$.

R_1	$v_I = v_{I1} + v_{I2}$	HS_Lower	FFT	MTA	AP	HS_Upper
0.01	0.0141	15.3370	15.7757	15.7537	15.7527	15.9080
0.02	0.0565	13.6258	15.0802	15.0445	15.0269	15.6341
0.03	0.1272	11.4519	13.9902	13.9531	13.8694	15.1854
0.04	0.2262	9.3072	12.5996	12.5897	12.3542	14.5728
0.05	0.3534	7.4385	11.0038	11.0697	10.5768	13.8106
0.055	0.4276	6.6314	10.1583	10.2837	9.6234	13.3787
0.06	0.5089	5.9078	9.2987	9.4948	8.6453	12.9159
0.063	0.5611	5.5115	8.7828	9.0239	8.0523	12.6245

7 Conclusions

There have been many previous studies on the elastic moduli of two-phase material composites or three-phase with periodic structures such as square or hexagonal models. In practical materials, the structure of materials is often randomly distributed and has multi-phases. In this work, with different asymptotic solutions, the paper has solved the problem of the elastic moduli for a three-phase material model in 2D.

FFT algorithm is developed to calculate the effective elastic moduli of some complex material models such as three-phase composites with an arbitrary distribution of two inclusion phases. The numerical results fall inside the Hashin-Shtrikman bounds.

DA, MTA approaches are to solve for the three-phase material model of two different sizes of circular inclusions. FFT, MTA, and DA give quite close results. All results satisfy HS bounds over all the volume proportions of the components.

MTA and DA have explicit algebraic expressions, so that easy to apply the estimates for the effective elastic moduli, hence might be more useful for engineers as first estimates of the effective elastic moduli of the composites.

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