Super-collimation of the radiation by a point source in a uniaxial wire medium

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Abstract – We investigate the radiation properties of a short horizontal dipole embedded in a uniaxial wire medium. It is shown that the uniaxial wire medium enables a super-collimation of the dipole radiation such that the radiation pattern has a singularity and the radiated fields are non-diffractive in the broadside direction. We derive a closed analytical formula for the power radiated by the dipole in the wire medium. Our theory demonstrates that as a consequencer of the ultrahigh density of photonic states of the nanowire array, the power radiated by the dipole is strongly enhanced as compared to the power that would be emitted in the dielectric host with no nanowires.

Key words: Wire medium, Effective medium theory, Super-collimation.

1 Introduction

The uniaxial wire medium – a periodic array of parallel metallic wires embedded in a dielectric host – is one of the most important and extensively studied metamaterials [1–9]. Nanowire materials have attracted considerable attention due to their peculiar electromagnetic properties, such as the strongly spatially dispersive (nonlocal) response [4–9] and an anomalously high density of photonic states [10–14]. These unusual properties are useful in many applications from microwave up to optical frequencies [9–26].

Numerous theoretical and analytical methods were developed in the last decade to accurately characterize the effective electromagnetic response of wire media [3–8]. Such tools make possible, for instance, the study of the wave propagation in wire medium slabs [7, 27–29] and solving source-free spectral problems for the natural modes in closed analytical form [30–32]. However, the study of the problem of radiation by localized sources embedded in a wire medium background was somehow on the back burner for a long time. Only recently this subject has been investigated in more detail [33–36]. In particular, the radiation properties of a short vertical dipole embedded in a uniaxial wire medium were investigated in reference [35], using both a nonlocal framework [4, 6] and a quasi-static approach relying on additional variables that describe the internal degrees of freedom of the medium [8]. The objective of this work is to further study the radiation problem of a short dipole embedded in a uniaxial wire medium. Specifically, we extend the analysis to the scenario wherein the dipole is horizontal with respect to the wires (see Figure 1) rather than vertical as in reference [35]. Importantly, we demonstrate that for a horizontal dipole the radiated fields can be super-collimated by the nanowires leading to a super-directive emission along the axial direction.

This paper is organized as follows. In Section 2, we introduce the radiation problem under study. In Section 3, we present the solution of the problem in the spectral domain using the nonlocal dielectric function approach. In Section 4, we derive a closed analytical formula for the power radiated by the short dipole in the metamaterial. Finally, in Section 5 the conclusions are drawn. Throughout this work we assume a time harmonic regime with time dependence of the form $e^{-i\omega t}$.

2 The radiation problem

Figure 1 illustrates the geometry of the problem under study: a square array of metallic wires embedded in a dielectric host. The spacing between the wires is $a$ and the wire radius is $r_w$. The excitation source is centered at the position $r' = (x', y', z')$ and corresponds to a short horizontal dipole described by the electric current density $j_{ext}(r) = -i\omega p_0 \delta(r - r') \hat{x}$, where $\omega$ is the angular frequency of oscillation, $p_0$ represents the electric dipole moment, and $\hat{x}$ is the unit vector along the positive x-axis. Our objective is to characterize the radiation emitted by the short dipole using an effective medium theory.

As a starting point, we remark that the use of effective medium methods requires that the source must be localized...
in a region with characteristic dimensions larger than the lattice constant $a$ of the metamaterial (see Figure 1), so that it is possible to assume that only waves with $-\pi/a \leq k_x, k_z \leq \pi/a$ can be excited in the metamaterial [35]. This property can be justified by the “uncertainty principle” of the Fourier transform, which establishes that the spreading of a function in the spatial and spectral domains is not independent, and the characteristic widths in the spatial ($\sigma_s$) and spectral ($\sigma_k$) domains are bound to satisfy $\sigma_s \sigma_k \geq \frac{1}{2}$. Therefore, if the characteristic width of a source is of the order of $\sigma_s \sim a$, one may assume $\sigma_k \sim 1/a$, which justifies taking a wave vector cut-off of the order $k_{\text{max}} \sim \pi/a$. Thus, even though we represent the dipole using the Dirac distribution $\delta$-symbol, in practice the dipole needs to have length/radius at least comparable to $a$, and should at the same time be much shorter than the wavelength. Such a current distribution can be modeled mathematically by

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3} \int_{-k_{\text{max}}}^{k_{\text{max}}} dk_x \int_{-k_{\text{max}}}^{k_{\text{max}}} dk_y \int_{-\infty}^{\infty} dk_z e^{i k \cdot \mathbf{r}}$$

being $k_{\text{max}}$ the spatial cut-off. Note that we do not need a spatial cut-off along the $z$-direction because the invariant is structureless trans formations along $z$. For $k_{\text{max}} \rightarrow \infty$ we recover the usual Dirac distribution.

It is known [4, 6] that the uniaxial wire medium formed by straight metallic wires is characterized by the effective dielectric function

$$\varepsilon(\omega, \mathbf{k}) = \frac{1}{\varepsilon_0 \varepsilon_\infty} \tilde{1} + \chi_{zz}(\omega, k_z) \hat{z} \hat{z},$$

where $\otimes$ denotes the tensor product of two vectors and $\varepsilon_\infty$ is the relative permittivity of the host material. In case of perfect electrical conducting (PEC) wires the $zz$ susceptibility is given by $\chi_{zz} = -\frac{k_p^2}{k_p^2 + \omega^2}$, where $k_p$ is the plasma wave number of the wire medium [4, 6], and $k_p = k_\parallel \sqrt{\varepsilon_\infty} = \omega \sqrt{\mu_0 \varepsilon_0 \varepsilon_\infty}$ is the wave number in the host region. The formula for $z_{zz}$ in the general case of lossy metallic wires can be easily obtained from reference [6]. Note that the effective dielectric function depends explicitly on $k_z \rightarrow -i \partial/\partial z$.

Next, based on this nonlocal framework, we calculate the fields radiated by the horizontal dipole in the unbounded uniaxial wire medium and derive an explicit formula for the radiated power.

3 The radiated fields

The strategy to determine the emitted fields is to solve the radiation problem in the spectral domain (i.e., in the Fourier spatial domain) and then, calculate the inverse Fourier transform to obtain the field distributions in the spatial domain.

It is known [35] that in a spatially dispersive medium, the Maxwell equations may be written in the space domain as follows:

$$\nabla \times \mathbf{E} = \omega \mu_0 \mathbf{H}, \quad (2)$$

$$\nabla \times \mathbf{H} = -i \omega \varepsilon_0 (\omega, -i \nabla) \cdot \mathbf{E} + \mathbf{j}_{\text{ext}}. \quad (3)$$

The dyadic operator $\varepsilon(\omega, -i\nabla)$ represents the effective dielectric function of the material and can be written explicitly as a convolution. In the spectral (Fourier) domain, we have the correspondence $-i\nabla \rightarrow \mathbf{k} = (k_x, k_y, k_z)$. Hence, after some straightforward manipulations of equations (2) and (3), one can find that the Fourier transform of the electric field satisfies [35]

$$\mathbf{E}(\omega, \mathbf{k}) = -i \omega \mu_0 \left\{ \mathbf{e}^z(\omega) \varepsilon(\omega, \mathbf{k}) + \mathbf{k} \otimes \mathbf{k} - k^2 \mathbb{I} \right\}^{-1} \cdot \mathbf{j}_{\text{ext}}, \quad (4)$$

with $\mathbf{j}_{\text{ext}} = -i \omega \mu_0 \mathbf{E}$. From equation (4), it follows that:

$$\mathbf{E}(\omega, k_j, k_z) = \frac{p_e}{\varepsilon_0 \varepsilon_\infty} \frac{1}{k_h^2 - k_j^2} \left[ \frac{k_h^2 - k_j^2 + k_z^2}{k_h^2 - k_j^2 + k_z^2} \right] \hat{x} + \frac{k_h^2 - k_j^2 + k_z^2}{k_h^2 - k_j^2 + k_z^2} \hat{y} + \frac{k_h^2 - k_j^2}{k_h^2 - k_j^2 + k_z^2} \hat{z} \right], \quad (5)$$

where $k_1 = \sqrt{k_h^2 + k_j^2}$ and $k_2 = k_h^2 + k_j^2$. For completeness, we mention that the radiated fields can also be written in terms of an electric Hertz potential $\Pi_e$ in the usual manner:

$$\mathbf{E}(\omega, \mathbf{k}) = \frac{\omega^2}{c^2} \varepsilon_0 \varepsilon_\infty \Pi_e - \mathbf{k} (\mathbf{k} \cdot \Pi_e), \quad (6)$$

$$\mathbf{H}(\omega, \mathbf{k}) = \omega \varepsilon_0 \varepsilon_\infty \mathbf{k} \times \Pi_e, \quad (7)$$

with

$$\Pi_e(\omega, \mathbf{k}) = \frac{1}{k_h^2 - k_j^2} \frac{p_e}{\varepsilon_0 \varepsilon_\infty} \left[ \mathbf{x} + \mathbf{z} \frac{k_h^2 - k_j^2 + k_z^2}{k_h^2 - k_j^2 + k_z^2} \right]. \quad (8)$$
In particular, the x-component of the Hertz potential can be explicitly integrated so that \( \Pi_{x,i}(\omega, r) = \frac{P_e}{\varepsilon_0 \varepsilon_r} \frac{e^{ik_r r}}{4\pi r} \) with \( r = \sqrt{x^2 + y^2 + z^2} \). Yet, for the analytical developments of the next section it is more useful to work directly with equation (5).

### 3.1 PEC nanowires

In the following, we focus the analysis on the particular case wherein the nanowires are PEC. In such a case, we get from equation (5):

\[
\mathbf{E}(\omega, k_x, k_z) = \frac{P_e}{\varepsilon_0 \varepsilon_r} \left[ \hat{x} \left( -\frac{k_h^2 k_h^2}{k_p^2 + k_h^2} \frac{1}{k_h^2 - k_z^2} \right) + \hat{y} \left( -\frac{k_h^2 k_h^2}{k_p^2 + k_h^2} \frac{1}{k_h^2 - k_z^2} \right) + \hat{z} \left( -\frac{k_h k_h}{k_p^2 + k_h^2} \frac{1}{k_h^2 - k_z^2} \right) \right].
\]

(9)

Interestingly, it is possible to calculate explicitly the inverse Fourier transform of the electric field in \( k_z \). A straightforward analysis shows that the components of the electric field satisfy:

\[
E_x(\omega, k_\parallel, z) = \frac{1}{2} \frac{P_e}{\varepsilon_0 \varepsilon_r} \left( i \frac{k_h^2 k_h^2}{k_p^2 + k_h^2} e^{ik_h z} - \frac{k_h^2}{\gamma_h} \frac{1}{k_p^2 + k_h^2} e^{-\gamma_h z} \right),
\]

(10)

\[
E_y(\omega, k_\parallel, z) = \frac{1}{2} \frac{P_e}{\varepsilon_0 \varepsilon_r} \left( i \frac{k_h^2 k_h^2}{k_p^2 + k_h^2} e^{ik_h z} - \frac{k_h^2}{\gamma_h} \frac{1}{k_p^2 + k_h^2} e^{-\gamma_h z} \right),
\]

(11)

\[
E_z(\omega, k_\parallel, z) = -i \frac{P_e}{\varepsilon_0 \varepsilon_r} \frac{k_h}{2} \frac{\text{sgn}(z)}{k_h} e^{-\gamma_h |z|},
\]

(12)

where \( \gamma_{TM} = \sqrt{k_h^2 + k_\parallel^2 - k_z^2} \) is the z-propagation constant of the transverse magnetic (TM) waves that propagate in the uniaxial wire medium, and \( \gamma_h = \sqrt{k_h^2 - k_z^2} \) is the z-propagation constant of the transverse electric (TE) waves. It is worth noting that each term of equations (10) and (11) is related to the contribution of each of the three eigenmodes supported by the uniaxial wire medium [4, 6, 8], namely the transverse electromagnetic (TEM) mode (term associated with the \( e^{ik_h z} \) spatial variation), the transverse electric (TE) mode (term associated with the \( e^{-\gamma_h |z|} \) spatial variation), and the transverse magnetic (TM) mode (term associated with the \( e^{-\gamma_{TM} |z|} \) spatial variation).

The most important radiation channel is associated with the TEM waves which have the propagation factor \( e^{ik_h z} \). The corresponding radiated fields can be explicitly integrated, and can be written in terms of an electric potential \( \phi \) as follows:

\[
\mathbf{E}_{\text{TEM}}(\omega, r) = -e^{ik_h z} \nabla \phi,
\]

(13)

with \( \nabla = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} \) and

\[
\phi(\omega, r) = -\frac{P_e}{2\varepsilon_0 \varepsilon_r} \frac{ik_h}{\rho} \frac{x}{\rho} - \frac{k_h \rho K_1(k_h \rho)}{2\pi \rho},
\]

(14)

where \( \rho = \sqrt{x^2 + y^2} \), and \( K_1 \) represents the modified Bessel function of the second kind and order n. This result is rather interesting, since it predicts that a non-diffractive cylindrical beam extremely confined to the wires axis can be excited inside the nanowire metamaterial. Note that the TEM beam does not decay with the distance as it propagates away from the point source, different from what happens in a standard dielectric wherein the fields decay as \( 1/r \). As a consequence, the radiation intensity of the horizontal dipole diverges to infinity for an observation direction parallel to the z-direction. Thus, the beam is super-collimated by the nanowires and the directivity of this elementary radiator diverges to infinity. It should be noted that the described behavior is fundamentally different from that found in reference [35] for a vertical dipole. Indeed, a vertical dipole is unable to excite the TEM waves in the nanowire material, and thus such an excitation cannot generate a diffractionless beam.

The total radiated field has also contributions from the excited TE and TM waves. For long wavelengths, \( k_h \ll k_\parallel \), the contribution of the TM mode is negligible in the far-field region because its attenuation constant is very large [29]. On the other hand, the TE waves lead to spherical wavefronts, but it will be shown later that they only transport a small fraction of the total radiated power, and hence in practice they are of secondary importance. Moreover, it can be analytically shown that in the \( xz \) plane the electric field associated with the TE mode decays as \( 1/r^2 \) along any observation direction, with the exception of the z-axis.

Notably, it may be verified that the field \( \mathbf{E}_{\text{TEM}} \) is singular along the \( z \)-axis such that \( E_z \) diverges logarithmically as \( \rho \to 0 \). This unphysical behavior is due to the fact that in the continuous limit (when \( -\infty < k_h, k_\parallel < \infty \)) the density of photonic states of the wire medium diverges [20, 22], and hence the metamaterial has infinite radiation channels leading to a spatial singularity of the radiated field. This problem can be easily fixed by introducing the spatial cut-off \( k_{\text{max}} = \pi / a \) such that the integration range in \( k_z \) and \( k_\parallel \) is truncated to
the first Brillouin zone, \(-\pi/a \leq k_x, k_y \leq \pi/a\). Indeed, the wave vector of the TEM modes must be restricted to the first Brillouin zone when the actual granularity of the metamaterial is properly considered [20]. The truncation of \(k_z\) to the first Brillouin zone is also consistent with our assumption that the point source is less localized than the period of the wire medium.

3.2 Numerical example

To illustrate the non-diffractive nature of the radiation transported by the TEM waves, we represent in Figure 2 the \(x\)-component of \(\mathbf{E}^{\text{TEM}}\) in the \(y = 0\) plane (the \(E\)-plane) and in the \(x = 0\) plane (the \(H\)-plane), obtained from equations (13) and (14). The lattice period is \(a\), the radius of the wires is \(r_w = 0.01a\), the host is a vacuum \(\varepsilon_h = 1\), and the frequency of operation is such that \(\omega a/c = 0.1\). The dipole is centered at the origin \((x, y, z) = (0, 0, 0))\).

\[ E_z(x, a) = \text{constant}, \quad y = 0 \]  
\[ H_y(x, a) = \text{constant}, \quad z = 0 \]

\[ E_x(x, a) = \text{constant}, \quad z = 0 \]  
\[ H_z(x, a) = \text{constant}, \quad y = 0 \]

\[ H_y(z, a) = \text{constant}, \quad y = 0 \]  
\[ E_x(z, a) = \text{constant}, \quad z = 0 \]

\[ H_z(y, a) = \text{constant}, \quad y = 0 \]  
\[ E_x(y, a) = \text{constant}, \quad z = 0 \]

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\[ H_y(z, a) = \text{constant}, \quad z = 0 \]  
\[ E_x(z, a) = \text{constant}, \quad z = 0 \]
characterized by a diffraction-free beam pattern in the far field (see Figure 4c).

4 The radiated power

Next, relying on the eigenwave expansion formalism introduced in reference [37], we derive a closed analytical formula for the power radiated by the short horizontal dipole inside the unbounded uniaxial wire medium (see Figure 1). To begin with, we present an overview of the eigenfunction expansion formalism. Then, we use this formalism to calculate the radiated power for PEC nanowires.

4.1 Overview of the eigenfunction expansion formalism

In reference [37], we derived a general analytical formulation that enables calculating the power emitted by moving sources in frequency dispersive lossless wire media. This formalism can be applied in a straightforward manner to standard non-moving sources, e.g. to an electric dipole. Specifically, using equation (11) of reference [37] and \( j_{\text{cut}} = -i \alpha_p \delta(\vec{r} - \vec{r}') \) it is simple to check that the electric field \( \vec{E} \) radiated by the point dipole in time-harmonic regime has the following exact modal expansion:

\[
\vec{E}(\vec{r}) = \frac{ \alpha p \varepsilon \phi }{ 2 V } \sum_n \frac{1}{\omega_n - \omega} \vec{E}_n(\vec{r}) \vec{E}^*_n(\vec{r}') \cdot \hat{x}.
\]

Here \( V = L_x \times L_y \times L_z \) is the volume of the region of interest, which in the end will be let to approach \( V \to \infty \). The symbol \( * \) denotes complex conjugation. The summation in equation (17) is over the electromagnetic (plane wave) modes of the bulk material \( (\vec{E}_n, \vec{H}_n) \), \( n = 1, 2, \ldots \), which for dispersive media must be normalized as follows [37]:

\[
\frac{1}{2} \frac{\partial}{\partial \omega} \left[ \omega \vec{E}(\omega, k) \right]_{\omega = \omega_n} \cdot \vec{E}_n + \frac{1}{2} \mu_0 \vec{H}_n \cdot \vec{H}_n = 1.
\]

The left-hand side of equation (18) is the stored energy density associated with a natural mode with time dependence \( e^{-i\omega t} \) [39, 40, 41]. The frequencies \( \omega_n \) are the real-valued eigenfrequencies of the natural modes. Importantly, the summation in equation (17) must include both the positive frequency and the negative frequency eigenmodes [37].

Since we are dealing with continuous media, it is clear that the eigenmodes \( \vec{E}_{n,k} \) can be taken as plane waves associated with a wave vector \( \vec{k} \). Hence, equation (17) becomes

\[
\vec{E}(\vec{r}) = \frac{ \alpha p \varepsilon \phi }{ 2 V } \sum_n \sum_k \frac{1}{\omega_n - \omega} \vec{E}_{n,k}(\vec{r}) \vec{E}^*_{n,k}(\vec{r}') \cdot \hat{x}.
\]
where \( \omega_{n,k} \) are the natural frequencies associated with the plane waves \( E_{n,k} \) with wave vector \( k \) and index \( n \) (\( n \) determines the eigenmode type). As discussed in reference [37], the eigenmodes \( E_{n,k} \) can be divided in four different types: TE modes, TM modes, TEM modes, and longitudinal \( (L) \) \( (\) electrostatic and magnetostatic) modes with \( \omega_{n,k} = 0 \). The LS modes do not contribute to the radiation field. However, they are mathematically important, since one cannot obtain a complete set of eigenfunctions without them [42].

In the continuous limit \( (V \rightarrow \infty) \), the summation over \( k \) is replaced by an integral and equation (19) becomes:

\[
E(r) = \frac{\alpha P}{16\pi^2} \sum_n \int d^3k \frac{1}{\omega_{n,k} - \omega} E_{n,k}(r) E^*_n(k) \cdot \hat{x}. \tag{20}
\]

4.2 The emitted power

The integrand of equation (20) is singular when \( \omega_{n,k} = \omega \), which corresponds to the isofrequency contours of the eigenmodes. Then, given that \( d^3k = ds(k)dk_j = ds(k)\partial \omega_{n,k}/|\nabla_k \omega_{n,k}| \) \( (ds(k) \) is the area of element of isofrequency surfaces), one can write equation (20) as follows:

\[
E(r) = \frac{\alpha P}{16\pi^2} \sum_n \int ds(k) \int d\omega \frac{1}{\omega_{n,k} - \omega} \times \frac{1}{|\nabla_k \omega_{n,k}|} E_{n,k}(r) E^*_n(k) \cdot \hat{x}. \tag{21}
\]

To avoid the singularity of the integrand we replace \( \omega \rightarrow \omega + \imath 0^+ \), such that the integration path in the upper-half frequency plane, consistent with the causality of the system response [37]. Then using the identity

\[
\frac{1}{\lambda - \imath 0^+} = \text{P.V} \frac{1}{\lambda} + \imath \nu \delta(\lambda) \tag{42},
\]

where P.V. denotes the Cauchy principal value, we may write equation (21) as follows:

\[
E(r) = \frac{\alpha P}{16\pi^2} \sum_n \int ds(k) \text{P.V.} \times \left( \int d\omega \frac{1}{\omega_{n,k} - \omega} \frac{1}{|\nabla_k \omega_{n,k}|} E_{n,k}(r) E^*_n(k) \cdot \hat{x} \right) + \frac{\alpha P}{16\pi^2} \sum_n \int_{\omega_{n,k} = 0} ds(k) \frac{1}{|\nabla_k \omega_{n,k}|} E_{n,k}(r) E^*_n(k) \cdot \hat{x}. \tag{22}
\]

In a time-harmonic regime the time-averaged radiated power is given by \( P_{\text{rad}} = -\frac{1}{2} \int d^3r \text{Re} \{ E \cdot \hat{x}_\text{ext}^* \} \), where \( E \) the macroscopic electric field and \( \hat{x}_\text{ext}^* = -\imath \alpha P \delta(r - r') \hat{x} \) the electric current density. The term associated with the principal value integral in equation (22) does not contribute to the radiated power, and hence \( P_{\text{rad}} \) can be written simply as:

\[
P_{\text{rad}} = \frac{\alpha^2 |P|^2}{32\pi^2} \sum_n \int_{\omega_{n,k} = 0} ds(k) \frac{|E_{n,k}(r) \cdot \hat{x}|^2}{|\nabla_k \omega_{n,k}|}. \tag{23}
\]

4.3 PEC nanowires

To illustrate the application of the theory, next we suppose that the metallic wires are perfect conductors. It is known that for long wavelengths \( (k_b \ll k_p) \) the only propagating modes in the uniaxial wire medium are the TE and the TEM modes [7, 12]. Hence, for long wavelengths the summation in equation (23) can be restricted to TE and TEM modes.

The dispersion characteristic of the (positive frequency) TEM eigenmodes is given by \( \omega = |k_z|c_0 \), where \( c_0 = c/\sqrt{\epsilon_0} \) [7]. On the other hand, the electric field of the TEM modes is of the form \( E_{\text{TEM},k} \sim \mathcal{A} k_z \), wherein \( \mathcal{A} \) is a normalization constant determined by equation (18). It was proven in reference [37] that \( A \) satisfies \( A = k_p/\sqrt{(k_z^2 + k_p^2)(k_z^2 + k_y^2 + k_p^2)} \epsilon_0 \epsilon_h \).

Substituting this result into equation (23), it is found that the power transported by the TEM modes is given by:

\[
P_{\text{rad,TEM}} = \frac{\alpha^2 |P|^2}{32\pi^2 c_0^2 \epsilon_h} \frac{k_p^2}{\epsilon_0 \epsilon_h} \int_0^{k_{\text{max}}} \int_0^{K_{\text{max}}} \frac{k^2}{(k_z^2 + k_p^2)(k_z^2 + k_y^2 + k_p^2)} dk d\phi. \tag{24}
\]

The integration range was restricted to the Brillouin zone because it is assumed that the point source is less localized than the lattice period \( (k_{\text{max}} = \pi/\alpha) \). Indeed, it is essential to include the spatial-cutoff in the calculation of the emitted power, otherwise it diverges. The leading factor of 2 follows from the fact that the isofrequency surfaces of the TEM waves are formed by two parallel planar sheets. For convenience, next we replace the integration over the square shaped Brillouin zone by an integration over a circular Brillouin zone with radius \( K_{\text{max}} = 2\pi/\alpha \), such that the integral (24) becomes:

\[
P_{\text{rad,TEM}} = \frac{\alpha^2 |P|^2}{32\pi^2 c_0^2 \epsilon_h} \frac{k_p^2}{\sin^2 \phi} \int_0^{2\pi} \int_0^{K_{\text{max}}} k^2 dk d\phi. \tag{25}
\]

Straightforward calculations prove that:

\[
P_{\text{rad,TEM}} = \frac{\alpha^2 |P|^2}{32\pi^2 c_0^2 \epsilon_h} \frac{k_p^2}{\epsilon_0 \epsilon_h} \ln \left( 1 + \frac{4\pi}{\alpha^2 k_p^2} \right). \tag{26}
\]

Equation (26) shows that the power radiated by the dipole inside the uniaxial wire medium increases as the separation between the wires \( a \) decreases. This can be understood as a consequence of the enhancement of the density of TEM modes when the distance between the wires becomes increasingly smaller. The density of photonic states for the TEM modes is \( \mathcal{D}_{\text{TEM}}(\omega) = \frac{1}{(2\pi)^2} \int_{\text{TEM,}\omega} ds(k) \frac{1}{|\nabla_k \omega_{\text{TEM}}|} = 1/(\pi a^2 c_0) \) [10, 11]. Note that from equation (18) we may estimate that \( |E_{\text{TEM}}(r) \cdot \hat{x}|^2 \sim \frac{1}{\epsilon_0 \epsilon_h} \). Using this approximation in equation (23) we get \( P_{\text{rad,TEM}} \approx \frac{\pi}{4} \frac{\alpha^2 |P|^2}{\epsilon_0 \epsilon_h} \mathcal{D}_{\text{TEM}}(\omega) \), which overestimates the result of equation (26) by a factor
of the emitted power is transported by TEM waves. Moreover, we derived a closed analytical expression for the power radiated by the dipole relying on an eigenfunction expansion [37]. It was demonstrated that, owing to a singularity in the density of photonic states of the uniaxial wire medium, the power radiated by the dipole is strongly enhanced as compared to the power emitted by the same dipole in the host dielectric. For realistic metallic wires the iso-frequency contours will become slightly hyperbolic and hence the radiated beam is expected to be slightly divergent. Finally, we note that when the length of the wire medium is finite along the z-direction the supercollimated beam will create a sharp near-field distribution with subwavelength features at the interface with an air region. Only the spatial harmonics with \( k_{il} < \alpha / c \) can be coupled to the propagating waves in free-space, and hence the rest of the energy will stay trapped in the wire medium slab.

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